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AN EXTENSION OF THE ALGEBRA OF LOGIC

BY the "algebra of logic" I mean, for the purposes of the present paper, Boole's calculus, in its classic form. I shall ignore, for the time, all the applications of this calculus to "classes," to "propositions," or to any other special sorts of objects. By the "Boolean entities" I shall here mean simply whatever entities conform to the laws which Boole's algebra expresses. There are such entities,—for instance, "classes," and "propositions," as well as "areas." But there are other Boolean entities than these; and I am here concerned only with Boolean entities abstractly viewed.

I

Considered as an algebra, the Boolean calculus, as is well known, occupies an unique but disappointing place among the algebras known to modern mathematics. In opening two remarkable contributions to the algebra of logic which were published a few years ago[1] Whitehead said that Boole's algebra might be compared with the chemical element argon. For, as Whitehead remarked, Boole's calculus had so far refused to form compounds with other equally elementary theories. Whitehead himself undertook, in the papers in question, to contribute towards the needed change of this unprofitable state of isolation. But, important as his results were, Whitehead has not, so far as I know, published further researches upon the same topic. The problem remains still on our hands: Can any one discover devices to make this argon among the pure algebras somewhat better disposed to unite with its fellow-algebras?

This question may seem to have a merely formal interest: and in this paper I shall deal wholly with formalities. Yet, as I hope to show in future papers, very interesting philosophical issues are bound up with the answer to the question which Whitehead's comparison of the Boolean calculus to argon presents to our notice. This is a region of philosophy where some of the most abstract, and some of the deepest and richest of philosophical interests lie very near together.

[1] See *American Journal of Mathematics,* Vol. 23, pages 139, 297.

II

The Nature of the Boolean Operations

The general reason why Boole's calculus has proved so austerely unproductive of the mathematical novelties for which Boole himself hoped, is well known. The fundamental operations of the Boolean calculus, *viz.*, the "addition" and the "multiplication" which characterize this algebra, appear, at first sight, to promise notable new combinations, since, like the corresponding operations in ordinary algebra, they are both commutative and associative. Furthermore, each of these Boolean operations is distributive with reference to the other. Their dual relation to "negation," as expressed in "De Morgan's theorem," is a very attractive property, which especially helps to give to this algebra its unique place amongst forms of symbolism. But alas, neither the "multiplication" nor the "addition" of this algebra is unconditionally invertible. Nor is the result of such inversion, when there is such a result, free from ambiguity. "Division" and "subtraction" have existence only subject to exasperating restrictions; and have never received any really notable development. In brief, the fundamental operations of the algebra are not group-operations. Hence the theory of groups is, except for one striking, but, so far, comparatively unfruitful exception, inapplicable to the Boolean algebra in its classic form. But without group-operations how shall an algebra progress?

III

The Operation of Jevons, the Prime-Functions of Whitehead, and the T-Relation

The exception just mentioned, the one group-operation which the Boolean calculus in its original form permits, was first noticed by Jevons. Schroeder and, still later, Whitehead, have dealt with it at considerable length. I have here only a word to add to the observations which Schroeder and Whitehead have made upon this particular topic; but this word relates, I believe, to a logical relation whose philosophical importance may appear, as I hope, in papers which I hope later to prepare.

Jevons noted that the three Boolean equations:—

$$a\bar{b}+\bar{a}b=c,$$
$$b\bar{c}+\bar{b}c=a,$$
$$a\bar{c}+\bar{a}c=b,$$

express precisely equivalent propositions, since each of them follows from either of the others. Schroeder and Whitehead have in somewhat different ways expressed the very natural observation that this

equivalence of the three equations of Jevons defines a group-operation.

Let $a\bar{b} + \bar{a}b = (a \circ b)$, where the symbol $\circ$ stands for an operation performed upon the elements of the pair (a, b). Then, if we suppose that this operation is viewed as already known,—we observe that whatever pair of the "logical entities" of the Boolean calculus we may choose, the entity $(a \circ b)$ is itself a perfectly determinate Boolean logical entity. The operation $\circ$ is commutative and associative. The three equations of Jevons show that this operation is not only invertible, but is also its own inverse, so that the three symbolic expressions, $(a \circ b) = c$, $(a \circ c) = b$, $(b \circ c) = a$, express mutually equivalent propositions. The group defined by this operation is the "axial group," as has been pointed out by Professor Miller, of the University of Illinois, to whom I owe this last observation.

Whitehead has proposed to call functions whose form is that of $a\bar{b} + \bar{a}b$, "prime functions" or simply "primes." For a reason, explained in his papers,[2] Whitehead also calls these functions "primary primes."

The equation: $a\bar{b} + \bar{a}b = c$ appears, at first sight, to be expressive of a triadic relation of the elements (a, b, c). But as early as 1905 I myself noted that the relation involved is in fact *tetradic.* This latter observation was at that time, I think, new; and I have never heretofore printed the very obvious statements regarding this tetradic relation which here follow, and which I have constantly used in lectures ever since 1905. Some fellow-students may still find novel the form of expression which I employ.

Let us suppose true the equation:—

$$a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d.$$

Then the four elements (a, b, c, d) stand in a symmetrical tetradic relation which can be expressed by solving the equation, successively, for a, for b, for c, and for d. Thus,

$$\begin{aligned} a &= (bc + \bar{b}\bar{c})d + (b\bar{c} + \bar{b}c)\bar{d}; \\ b &= (ad + \bar{a}\bar{d})c + (a\bar{d} + \bar{a}d)\bar{c}; \\ &= (ac + \bar{a}\bar{c})d + (a\bar{c} + \bar{a}c)\bar{d}. \end{aligned}$$

That is, in case the equation: $a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d$, is true, each of the four elements of the tetrad (a, b, c, d) is a determinate symmetrical function of the three remaining elements, while the form of this function remains the same, whatever one of the four elements we choose to express in terms of the others. I propose to symbolize the totally symmetrical tetradic relation here in question by the relational form $T(abcd)$, which is to be read as a proposition: "The four

[2] See *American Journal of Mathematics,* Vol. 23, page 147.

elements (a, b, c, d) stand in the four-term relation T.'' The proposition $T(abcd)$ is thus precisely equivalent to any one of the equations:

$$ab + \bar{a}b = c\bar{d} + \bar{c}d; \quad a\bar{c} + \bar{a}c = b\bar{d} + \bar{b}\bar{d}, \text{ etc.}$$

The properties of this T-relation are then the following:—

1. The relation is (as just pointed out) totally symmetrical. That is, $T(abcd) = T(cdba) = T(dcab)$, etc.

2. Given (a, b, c) or, in fact, given any triad of the elements of the tetrad (a, b, c, d), chosen quite without restriction,—then, by the requirement that $T(abcd)$ shall be true, the fourth element of the tetrad is uniquely determined.

3. The T-relation $T(abcd)$ remains invariant in case, for any *two* of the elements of the tetrad, we substitute their respective negatives. Thus $T(abcd) = T(ab\bar{c}\bar{d}) = T(\bar{a}b\bar{c}d)$, etc.

4. The T-relation is *transitive by pairs*. That is, if $T(abcd)$, and if $T(abef)$, it follows that $T(cdef)$. For $a\bar{b} + \bar{a}b = c\bar{d} + \bar{c}d = e\bar{f} + \bar{e}f$.

5. If $T(abcd)$ and if $a = b$, then it follows that $c = d$; if $b = c$, then $a = d$, etc. Conversely, $T(aabb)$, $T(\bar{a}\bar{a}bb)$, $T(aaaa)$, are true propositions, whatever Boolean elements a and b may be.

If, hereupon, we make the logical element 0 a member of a tetrad, and if $T(abc0)$ is true, this fact may also be expressed by any one of the three equations of Jevons:—

$$a\bar{b} + \bar{a}b = c, \quad b\bar{c} + \bar{b}c = a, \quad a\bar{c} + \bar{a}c = b.$$

It follows that equations involving the ''prime-functions'' of Whitehead can be expressed in terms of T-relations; and that, in particular, the equation: $a\bar{b} + \bar{a}b = c$ should be regarded, *not* as simply expressive of a triadic relation of (a, b, c), but rather as expressing a symmetrical tetradic relation of a, b, c, and 0.

This, the only group-operation of the classical Boolean algebra, may thus be regarded as deriving its properties from those of the T-relation. The total symmetry of this tetradic relation is responsible for the simplicity of the group in question, and for its comparative unfruitfulness as a source of novelties in the Boolean calculus.

I shall have occasion to return to the T-relation in various stages of the inquiry which is herewith begun.

IV

The Ordinal Functions

So far we have traversed what is, on the whole, decidedly familiar ground. From this point onwards we shall deal with matters which I believe to be novel.

There exists a class of functions, in the algebra of logic, to which, since 1909, when I first observed some of their more interesting properties, I have devoted a good deal of study. Here is not yet the place to tell why these functions are as important for logical theory as I believe them to be. My present task will be limited to defining these functions, and to showing that, by making a due use of them, we can easily lay the foundation for a very considerable extension of the classical Boolean calculus. I shall give to the functions in question the name "ordinal functions."

Every student of Boole's algebra is familiar with functions of the form:—

$$p = axy + bx\bar{y} + c\bar{x}y + d\bar{x}\bar{y}.$$

It occasionally happens, in dealing with the solution of equations, that one may meet with the special case where $a = \bar{d}$, $b = \bar{c}$. The foregoing function then becomes:—

$$\begin{aligned} p &= axy + bx\bar{y} + \bar{b}\bar{x}y + \bar{a}\bar{x}\bar{y} \\ &= abx + \bar{a}\bar{b}\bar{x} + a\bar{b}y + \bar{a}b\bar{y}. \end{aligned}$$

If $p = 0$, the resulting equation has an interesting pair of roots. For by virtue of the foregoing transformation, which the special function here in question permits, the two unknowns (x, y) in the equation:—

$$axy + bx\bar{y} + \bar{b}\bar{x}y + \bar{a}\bar{x}\bar{y} = 0,$$

can be separated for the purpose of solution. For we have, as above:

$$abx + \bar{a}\bar{b}\bar{x} + a\bar{b}y + \bar{a}b\bar{y} = 0.$$

Hence

$$\begin{aligned} \bar{a}b &\prec x \prec \bar{a} + \bar{b}, \\ \bar{a}b &\prec y \prec \bar{a} + b. \end{aligned}$$

The equation is therefore soluble, not only without condition, but in a peculiarly simple form.

By an ordinal function I mean a function of four terms which has the form here illustrated by p.[3]

In the foregoing case, p appears as a function of the two unknowns (x, y), and of the two coefficients (a, b), together with their negatives $(\bar{a}, \bar{b})$. For our further purposes, this distinction between the "unknowns" and "the coefficients" may be left out of account. But we still need to note that an ordinal function is to be viewed as a function of four terms which are regarded as first divided into two pairs, while the function itself depends upon the way in which these pairs are chosen, are arranged, and then are submitted to the operations which characterize the form above illustrated.

[3] The ordinal functions here defined form one special case in the class of functions which are sums or products of Whitehead's "secondary primes." But Whitehead does not discuss in any detail this special case.

A few examples may help to make clear what is essential to the form of an ordinal function. Let the four elements (a, b, c, d) and their respective negatives be used. Let

$$
\begin{aligned}
p &= abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d},\\
q &= a\bar{b}c + \bar{a}b\bar{c} + a\bar{c}d + \bar{a}c\bar{d},\\
r &= b\bar{c}d + \bar{b}c\bar{d} + ab\bar{d} + \bar{a}\bar{b}d,\\
s &= \bar{a}\bar{c}\bar{d} + acd + abc + \bar{a}\bar{b}c.
\end{aligned}
$$

In the formation of each of these functions there is followed a set of rules which may be grasped by analyzing first the structure of the function p.

In order to form p we first select the two pairs (a, b) and (c, d). Let the pair (a, b) be called, for the moment, the *first* of these pairs, and the pair (c, d) the *second* pair. Each of these pairs, as it is here written, has its own first and its own second member. p is then defined by means of the five directions which here follow:

1. Take the continued product of both elements of the first pair into the first element of the second pair. We thus get the product abc.

2. Then form the product of the respective negatives of both elements of the first pair into the negative of the first member of the second pair. We now have the product $\bar{a}\bar{b}\bar{c}$.

3. Next form the continued product of the first member of the first pair, into the negative of the second member of the first pair, and into the second member of the second pair. We thus form the product abd.

4. Hereupon form the product of the negative of the first member of the first pair, into the second member of the first pair, and into the negative of the second member of the second pair. This gives us the product $\bar{a}b\bar{d}$.

5. Finally, take the sum of the four products thus successively formed. This sum gives us $g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d}$.

It will facilitate a survey of the rule whereby an ordinal function is formed, to use the following new symbolism. Let

$$abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d} = (ab\,;\, cd).$$

Here the expression which follows the symbol $=$ is a new symbol introduced, quite arbitrarily, simply to remind us of the foregoing five directions. This new symbol has the advantage of showing that the pairs (a, b) and (c, d) are distinguished, for the purpose here in question, as the first and the second pair. In the arbitrary symbol $(ab\,;\, cd)$, the semicolon separates these pairs. And the expressions ab, and cd, when thus separated by a semicolon, and together inserted, in this order, in a parenthesis, are not, in that context, to be understood as symbolizing logical products. The ex-

pression: $g = (ab;\ cd)$ is to be read: "The function g is that function of the four elements (a, b, c, d), which is formed by taking (a, b) and (c, d) as distinct pairs, whereof (a, b) is the first pair and (c, d) the second, and by then applying the five directions given above. Since these directions are themselves quite general, it would be easy to interpret as determinate ordinal functions any analogous expression, such as $x = (pq;\ rs)$; $y = (pr;\ qs)$, etc. Thus $x = (pq;\ rs)$ is equivalent, in terms of the ordinary Boolean calculus, to the equation:

$$x = pqr + \bar{p}\bar{q}\bar{r} + p\bar{q}s + \bar{p}q\bar{s}.$$

And $y = (pr;\ qs)$ is equivalent to the equation:—

$$\begin{aligned} y &= prq + \bar{p}\bar{r}\bar{q} + p\bar{r}s + \bar{p}r\bar{s} \\ &= pqr + \bar{p}\bar{q}\bar{r} + p\bar{r}s + \bar{p}r\bar{s}. \end{aligned}$$

As to the following functions of the tetrad (a, b, c, d):

$$\begin{aligned} h &= abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc, \\ m &= a\bar{b}c + \bar{a}b\bar{c} + a\bar{c}d + \bar{a}c\bar{d}, \end{aligned}$$

it is plain that they may be written:

$$\begin{aligned} h &= (ab;\ dc) = (a\bar{b};\ \bar{c}d), \\ m &= (ac;\ \bar{b}d) = (a\bar{c};\ d\bar{b}). \end{aligned}$$

For if we choose any one of the four expressions here written in parentheses, and if we regard each expression as a shorthand direction to apply our rule for forming ordinals to the two pairs which the semicolon (as used in each of these cases), separates, the resulting functions are as we have just written them.

In sum then, I mean, by an ordinal function, a function of four elements such that, if we begin with the form $(ab;\ cd)$, interpreted as above, we can form all possible ordinal functions by substituting for the tetrad (a, b, c, d) other tetrads of symbols which stand for logical elements; by changing the order of the elements and of the pairs which are in question; and by introducing the negatives, $\bar{a}$, $\bar{b}$, etc., into the expressions.

In case we confine ourselves to the elements (a, b, c, d) and their respective negatives, the permutations and arrangements possible in defining the forms $(ab;\ cd)$, $(\bar{a}\bar{b};\ \bar{d}c)$ $(ca;\ db)$ $(a\bar{d};\ \bar{b}c)$, etc., appear, at first sight, too numerous for an easy survey. But there are relations between pairs of formally distinct ordinal functions which greatly simplify the task of following the variations that our definition permits. Of these relations, the most important is expressed by the symbolic equation:

$$(ab;\ cd) = (cd;\ ab).$$

In the ordinary notation this becomes:

$$abc + \bar{a}\bar{b}\bar{c} + abd + \bar{a}b\bar{d} = acd + \bar{a}\bar{c}\bar{d} + bc\bar{d} + \bar{b}\bar{c}d.$$

The verification of this latter equation requires only a very simple computation. But the property expressed is especially characteristic of the ordinal functions. In view of the solution above given in case of an equation whose left-hand member is an ordinal function, it may require but a little reflection to see that the transitivity of the illative relation, the theory of elimination, as it forms part of the theory of logical equations, and consequently the ordinary theory of the syllogism, may all of them be viewed as standing in a close relation to this fundamental principle of the theory of the ordinal functions.

In consequence of the foregoing, we may now readily verify the symbolic equations

$$(ab\,;\,cd) = (cd\,;\,ab) = (a\bar{b}\,;\,dc) = (\bar{a}\bar{b}\,;\,\bar{c}\bar{d}) = (cd\,;\,ba) = (\bar{c}\bar{d}\,;\,\bar{a}\bar{b}).$$

The negation of ordinal functions leads to interesting forms.

If

$$g = (ab\,;\,cd),\ \text{then}\ \bar{g} = (ab\,;\,\bar{c}d) = (cd\,;\,\bar{a}b).$$

For

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d};$$

and therefore,

$$\bar{g} = ab\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{d} + \bar{a}bd;$$

while, by the foregoing,

$$g = acd + \bar{a}\bar{c}\bar{d} + bc\bar{d} + \bar{b}\bar{c}d;$$

and therefore,

$$\bar{g} = \bar{a}cd + a\bar{c}\bar{d} + \bar{b}c\bar{d} + b\bar{c}d.$$

When we study the ordinal functions which may be derived from $(ab\,;\,cd)$ by changing the order of the elements or of the pairs, and by introducing, in various possible ways, the symbol for negation, we find cases where two such functions are not, in general, equivalent, but become so upon condition that the elements of the tetrad (a, b, c, d) stand in some definite relation to one another. Thus let $g = (ab\,;\,cd)$, while $h = (ba\,;\,dc)$.

Then

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d},$$
$$h = abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc = (ab\,;\,d\bar{c}).$$

Hence,

$$\bar{g} = ab\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{d} + \bar{a}bd = (ab\,;\,\bar{c}\bar{d}),$$
$$\bar{h} = ab\bar{d} + \bar{a}\bar{b}d + a\bar{b}c + \bar{a}b\bar{c} = (ab\,;\,dc).$$

In consequence,

$$g\bar{h} + \bar{g}h = abc\bar{d} + ab\bar{c}d + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + \bar{a}\bar{b}cd + a\bar{b}\bar{c}\bar{d} + \bar{a}bcd + \bar{a}b\bar{c}\bar{d}.$$

Or, more briefly stated,

$$g\bar{h} + \bar{g}h = (ab + \bar{a}\bar{b})(c\bar{d} + \bar{c}d) + (a\bar{b} + \bar{a}b)(cd + \bar{c}\bar{d}).$$

If $g=h$, then both the first and the second members of this equation reduce to zero. In that case we have: $a\bar{b}+\bar{a}b=c\bar{d}+\bar{c}d$. This last expression is equivalent to asserting $T(abcd)$.

Therefore, if we transform the ordinal $g=(ab;\ cd)$ into the ordinal $(ba;\ dc)=h$, by separately reversing the order of the members of the first and the second pairs of the symbol for g, while leaving all else unchanged, then the ordinals g and h can be mutually equivalent if, and only if, $T(abcd)$. The T-relation thus stands in close connection with the properties of the ordinal functions.

These few cases serve to give a first glimpse of the decidedly interesting properties of the ordinal functions. I omit further developments at this stage of the inquiry; but I hope to have more to say about the ordinals in a subsequent paper.

V

Directed Pairs, and an Operation upon Such Pairs

We are now ready to define an operation which is based upon the operations of the ordinary algebra of logic, but which is applied to a new system of logical entities.

These new entities are *pairs* of Boolean logical elements. Just as, in the familiar modern theory of the rational numbers, a rational number is defined as a pair of whole numbers, so, in the theory to which I now pass, the entities to be considered are not Boolean elements, such as a, b, c, etc., but are pairs of such elements. And precisely as, in the modern theory of the rational numbers, the pairs of natural numbers which are in question are *directed* pairs, each pair having its first and its second member, or its upper and its lower member, so, in the following theory of logical pairs, I shall deal with directed logical pairs. Let the symbol a/b, or $\frac{a}{b}$, be used for the directed pair consisting of the elements (a, b), the element a being taken as the first, and b as the second member.

Our foregoing sketch of the properties of the ordinals has called our attention to the pairs of elements which are used in defining such an ordinal as $g=(ab;\ cd)$. But the pairs of which we made mention in our foregoing account of the ordinals, were not treated *as* pairs. The elements used entered, in a determinate way, into the definition of g. But g was itself a Boolean element. The functions in questions were Boolean functions. The elements a, b, c, etc., were treated only in so far as they were subjected to the operations of the Boolean calculus. The symbol $(ab;\ cd)$ was itself a mere shorthand for expressing the rule whereby the formation of an ordinal function was guided.

But from this point onwards we are to deal with our directed pairs as new entities, and are to subject them to new operations.

VI

The Ordinal Pair-Operation

Let a/b, c/d be the symbols, respectively, of the two directed pairs (a, b), (c, d). The form a/b means simply that a is taken as the first, and b as the second member of the pair (a, b), for the purpose of the operations into which these pairs are to enter. The symbol c/d has an analogous interpretation in case of the pair (c, d). This adoption of the symbols which are ordinarily used for fractions has in this case *merely* the significance which is given to it by the definition just stated.

Now the two directed pairs a/b and c/d may be so combined as to define, uniquely, a new pair. This pair shall be defined as the directed pair of ordinals g/h, when $g=(ab\,;\,cd)$ and $h=(ba;\,dc)$. Let the combination in question be viewed as an operation upon the pairs a/b, c/d. Symbolize this operation by $\circ$. Then

$$\frac{a}{b}\circ\frac{c}{d}=\frac{g}{h}=\frac{(ab\,;cd)}{(ba\,;dc)}.$$

It is plain that the directed pair g/h is uniquely determined by the directed pairs a/b, c/d.

Since $(ab;\,cd)=(cd;\,ab)=g$, while $(ba;\,dc)=(dc;\,ba)=h$, the operation $\circ$, as now defined, is commutative. That is, $a/b\circ c/d=c/d\circ a/b$.

Hereupon, we come to that feature of the new operation which constitutes the first contrast by which it is distinguished both from the addition and from the multiplication of the ordinary Boolean calculus. The new operation, namely, is *invertible*. That is, given the pair g/h, and either of the pairs a/b or c/d, the other of these pairs is uniquely determined.

For from the equations

$$\begin{aligned} g&=abc+\bar a\bar b\bar c+a\bar b d+\bar a b\bar d\\ &=acd+\bar a\bar c\bar d+bc\bar d+\bar b\bar c d,\\ h&=abd+\bar a\bar b\bar d+a\bar b\bar c+\bar a bc\\ &=bcd+\bar b\bar c\bar d+\bar a c\bar d+a\bar c d \end{aligned}$$

we can deduce, by an easy computation, the consequences:—

$$\begin{aligned} a&=cdg+\bar c\bar d\bar g+c\bar d\bar h+\bar c dh=(cd;\,g\bar h)=(dc;\,gh),\\ b&=cdh+\bar c\bar d\bar h+c\bar d g+\bar c d\bar g=(cd;\,h\bar g)=(hg;\,cd),\\ c&=abg+\bar a\bar b\bar g+a\bar b\bar h+\bar a bh=(ab;\,g\bar h)=(ba;\,gh),\\ d&=abh+\bar a\bar b\bar h+a\bar b g+\bar a b\bar g=(ab;\,hg)=(hg;\,ab). \end{aligned}$$

The interest of these facts is greatly increased by the consideration

that the invertibility of our operation is possible universally and without restriction, and can be expressed in a peculiarly simple form. Thus, if

$$a/b \circ c/d = g/h,$$

it is easy to show that

$$g/h \circ b/a = c/d,$$

while

$$g/h \circ d/c = a/b.$$

This, in fact, is precisely what is formulated by means of the four equations written above, and expressive of the values of a, b, c, and d.

The novelty and simplicity of the considerations here involved, make it worth while to express the facts in the form of what may be called a "Newlin-diagram,"[4] which brings before the eye the divisions of a universe of discourse containing four independent terms. It is well to remember, however, that any such diagram expresses a special application of the Boolean calculus, while the laws here in question hold true of the pure algebra, and are independent of all applications to "classes," to "areas," or to other special sorts of entities.

		a		$\bar{a}$		
		$a\bar{b}$	ab	$\bar{a}b$	$\bar{a}\bar{b}$	
c	$c\bar{d}$	$\bar{g}\bar{h}$	$g\bar{h}$	gh	$\bar{g}h$	
	cd	$g\bar{h}$	gh	$\bar{g}h$	$\bar{g}\bar{h}$	d
$\bar{c}$	$\bar{c}d$	gh	$\bar{g}h$	$\bar{g}\bar{h}$	$g\bar{h}$	
	$\bar{c}\bar{d}$	$\bar{g}h$	$\bar{g}\bar{h}$	$g\bar{h}$	gh	
			b			

Let

$$g = abc + \bar{a}\bar{b}\bar{c} + abd + \bar{a}b\bar{d} = (ab\,;\ cd),$$
$$h = abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc = (ba\,;\ dc).$$

The inverse operation just noted can easily be read off from this diagram. The formal analogy of our new operation, and of its inverse, to the multiplication and division of ordinary rational numbers, is, as far as it goes, worthy of notice. In case of the rational

[4] See the article by Professor W. J. Newlin, in this JOURNAL, Vol. III., page 539.

numbers, to divide by a rational number is equivalent to multiplying by the inverse, that is, by the reciprocal of the pair of whole numbers which constitutes the divisor. Our pairs of Boolean entities obey, with respect to our new operation, an analogous rule. The inverse of our operation is the direct operation with one of the factors of the original operation inverted, and then combined with the result of the original operation.

VII

The Associative Property of the Ordinal Pair Operation

We next come to a still more important fact. Our new operation is not only commutative and invertible, but also associative. That is, if we have given to us the three independent pairs a/b, c/d, e/f, then

$$\left[\frac{a}{b}\circ\frac{c}{d}\right]\circ\frac{e}{f}=\frac{a}{b}\circ\left[\frac{c}{d}\circ\frac{e}{f}\right].$$

The computations needed to establish this property of our operation are of necessity a little diffuse for complete statement in so summary a paper as the present one. It may be worth while, however, to give an outline of what seems to be a convenient mode of dealing with the matter, leaving to the reader the verification of the details of the computation, if he chooses to work them out. It may be remarked that, by means of a six-term "Newlin-diagram," or better by means of a pair of such diagrams, each diagram presenting to the eye one of the associations of pairs which is in question, the associative property of our operation can be made visible. But a six-term Newlin-diagram is somewhat troublesome to print.

It is worth noting, and is easily verifiable, that, if $a/b \circ c/d = g/h$, then

$$\begin{aligned}
gh &= abcd + \bar{a}bc\bar{d} + a\bar{b}\bar{c}d + \bar{a}\bar{b}\bar{c}\bar{d} = (ad + \bar{a}\bar{d})(bc + \bar{b}\bar{c}),\\
g\bar{h} &= abc\bar{d} + a\bar{b}cd + \bar{a}b\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}d = (ac + \bar{a}\bar{c})(b\bar{d} + \bar{b}d),\\
\bar{g}h &= a\bar{b}\bar{c}\bar{d} + ab\bar{c}d + \bar{a}bcd + \bar{a}\bar{b}c\bar{d} = (a\bar{c} + \bar{a}c)(bd + \bar{b}\bar{d}),\\
\bar{g}\bar{h} &= a\bar{b}c\bar{d} + ab\bar{c}\bar{d} + \bar{a}b\bar{c}d + \bar{a}\bar{b}cd = (b\bar{c} + \bar{b}c)(a\bar{d} + \bar{a}d).
\end{aligned}$$

All this may be read off directly from the Newlin diagram printed above. Hereupon we may set $g/h \circ e/f = r/s$.

That is,

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f}=\frac{r}{s}.$$

Now, by the definition of our operation,

$$\frac{g}{h}\circ\frac{e}{f}=\frac{r}{s}=\frac{ghe+\bar{g}\bar{h}\bar{e}+g\bar{h}f+\bar{g}h\bar{f}}{ghf+\bar{g}\bar{h}\bar{f}+gh\bar{e}+\bar{g}he}.$$

Substituting, in the right-hand member of this equation, the values of gh, $g\bar{h}$, $\bar{g}h$, and $\bar{g}\bar{h}$, as given above, we find:

$$r = (ad + \bar a\bar d)(bc + \bar b\bar c)e + (a\bar d + \bar a d)(b\bar c + \bar b c)\bar e$$
$$+ (ac + \bar a\bar c)(b\bar d + \bar b d)f + (a\bar c + \bar a c)(bd + \bar b\bar d)\bar f,$$
$$s = (ad + \bar a\bar d)(bc + \bar b\bar c)f + (a\bar d + \bar a d)(b\bar c + \bar b c)\bar f$$
$$+ (a\bar c + \bar a c)(bd + \bar b\bar d)e + (ac + \bar a\bar c)(b\bar d + \bar b d)\bar e.$$

But the directed pair r/s is, as we have defined it, the equivalent of

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f}.$$

Hereupon let us set

$$\frac{c}{d}\circ\frac{e}{f} = \frac{m}{n},$$

and suppose

$$\frac{a}{b}\circ\frac{m}{n} = \frac{u}{v}.$$

Then, by our definition,

$$\frac{a}{b}\circ\left(\frac{c}{d}\circ\frac{e}{f}\right).$$

We have also

$$\frac{m}{n} = \frac{(cd\,;\ ef)}{(dc\,;\ fe)} = \frac{cde + \bar c\bar d\bar e + c\bar d f + \bar c d\bar f}{cdf + \bar c\bar d\bar f + c\bar d\bar e + \bar c de}.$$

Furthermore

$$\frac{a}{b}\circ\frac{m}{n} = \frac{u}{v}.$$

Hence

$$u = abm + \bar a\bar b\bar m + a\bar b n + \bar a b\bar n.$$

Substituting the values of m and of n, respectively, as defined above, we discover hereupon that

$$u = ab(cde + \bar c\bar d\bar e + c\bar d f + \bar c d\bar f)$$
$$+ a\bar b(cdf + \bar c\bar d\bar f + c\bar d\bar e + \bar c de)$$
$$+ \bar a b(cd\bar f + \bar c\bar d f + c\bar d e + \bar c d\bar e)$$
$$+ \bar a\bar b(cd\bar e + \bar c\bar d e + c\bar d\bar f + \bar c df).$$

Rearranging this expression for the value of u, we find:

$$u = e(ad + \bar a\bar d)(bc + \bar b\bar c) + \bar e(a\bar d + \bar a d)(b\bar c + \bar b c)$$
$$+ f(ac + \bar a\bar c)(b\bar d + \bar b d) + \bar f(a\bar c + ac)(bd + \bar b\bar d).$$

Comparing this result with the value above given for r, we find that $r = u$.

By a precisely analogous computation we find that $s = v$, and thus we reach the result that:

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f} = \frac{a}{b}\circ\left(\frac{c}{d}\circ\frac{e}{f}\right) = \frac{r}{s} = \frac{u}{v}.$$

Our operation is therefore not only invertible and commutative,

but also associative. Accordingly, with respect to our new operation, the directed pairs of Boolean entities constitute an Abelian group.

VIII

Results

The properties of this group of Boolean pairs are sufficiently remarkable to deserve a summary statement. Each of the following propositions can easily be verified. The computations involved are, on the basis of the foregoing, all of them exteremely simple. They are also, so far as I know, novel.

The system of pairs now defined possesses a unit. This unit is the pair 1/1. For, by definition,

$$\frac{a}{b} \circ \frac{1}{1} = \frac{ab1 + \bar{a}\bar{b}\bar{1} + a\bar{b}1 + \bar{a}b\bar{1}}{ab1 + \bar{a}\bar{b}\bar{1} + a\bar{b}\bar{1} + \bar{a}b1} = \frac{ab + a\bar{b}}{ab + \bar{a}b} = \frac{a}{b}.$$

That is, whatever pair a/b be combined with the pair 1/1, is left invariant by that combination under the rules of our operation.

Owing to the formal analogy between our ordinal pair-operation and the multiplication of rational numbers (that analogy, namely, of which we made mention above, in speaking of the inverse of our operation), we may treat our ordinal pair-operation as a multiplication, although I have, at present, no addition-operation to set side by side with it. Regarding our operation, then, as a multiplication, we may write:

$$\frac{a}{b} \circ \frac{a}{b} = \left(\frac{a}{b}\right)^2.$$

We may use similar expressions for higher powers of a/b, and speak of cubes, etc. Only here we quickly find ourselves limited by the interesting further group-properties of our pair-operation which may next be stated.

We have, namely:

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \circ \frac{a}{b} = \frac{aba + \bar{a}\bar{b}\bar{a} + a\bar{b}b + \bar{a}b\bar{b}}{abb + \bar{a}\bar{b}\bar{b} + a\bar{b}\bar{a} + \bar{a}ba}.$$

Hence

$$\left(\frac{a}{b}\right)^2 = \frac{ab + \bar{a}\bar{b}}{ab + \bar{a}\bar{b}}.$$

The square of any pair is, therefore, a pair consisting of equal terms. Each of these equals is a "prime-function," *viz.*, the function, $ab + \bar{a}\bar{b}$, of the members of the pair.

We therefore have, for the cube of any pair, the expression:

$$\left(\frac{a}{b}\right)^3 = \frac{ab + \bar{a}\bar{b}}{ab + \bar{a}\bar{b}} \circ \frac{a}{b} = \frac{(ab + \bar{a}\bar{b})a + (a\bar{b} + \bar{a}b)\bar{a}}{(ab + \bar{a}\bar{b})b + (a\bar{b} + \bar{a}b)\bar{b}} = \frac{ab + \bar{a}b}{ab + a\bar{b}} = \frac{b}{a}.$$

That is to say, the cube of any pair is the inverse of that pair. Or, if the inverse of our "multiplication" be regarded, for the present purpose, as a "division," we now observe that to "multiply" by the cube of any pair is equivalent to multiplying by the inverse of that pair, or is, in other words, equivalent to dividing by that pair. In still another expression our result is that, if $a/b \circ c/d = g/h$, then $(a/b)^3 \circ g/h = c/d$; while $g/h \circ (c/d)^3 = a/b$. We next proceed to the fourth powers of pairs. We have:

$$\left(\frac{a}{b}\right)^4 = \frac{a}{b} \circ \left(\frac{a}{b}\right)^3 = \frac{a}{b} \circ \frac{b}{a} = \frac{1}{1}.$$

The last of these equations is reached as follows:

$$\frac{a}{b} \circ \frac{b}{a} = \frac{abb + \bar{a}\bar{b}\bar{b} + a\bar{b}a + \bar{a}b\bar{a}}{aba + \bar{a}\bar{b}\bar{a} + a\bar{b}\bar{b} + \bar{a}bb} = \frac{ab + \bar{a}\bar{b} + a\bar{b} + \bar{a}b}{ab + \bar{a}\bar{b} + a\bar{b} + \bar{a}b}.$$

Hereupon, we observe that

$$\left(\frac{a}{b}\right)^5 = \left(\frac{a}{b}\right)^4 \circ \frac{a}{b} = \frac{a}{b} \circ \frac{1}{1} = \frac{a}{b}.$$

Thus the fourth power of every pair is the unit pair, while the fifth power of each pair is identical with the pair itself. The "period" or "order" of our pair-operation is five. If we conceive the system of directed Boolean pairs as transformed within itself by combining each pair with itself by means of the pair-operation, and by then passing to the higher powers, the first such transformation substitutes for each pair its square, a determinate pair of equals, whose members are each of them a determinate "prime-function" of the original pair. Next, the "cubes," which are the inverses of the original pairs, are produced. The next such transformation substitutes for each and all the pairs the unit pair. Combining this unit pair with each member of the original system leaves that system invariant.

IX

Modulus Pairs

The pair 1/1, the unit pair, may be called the modulus of our ordinal pair-operation. It is evident that, whatever element a may be, the equation $(a/a)^2 = 1/1$ is always true. That is, the modulus is the square of any pair that consists of equal Boolean elements.

Our system of directed pairs contains, however, other pairs which have the properties of moduli; for each such modulus pair may be regarded as the unit of an ordinal pair-operation whose group is the same as the one which we have just been studying, and whose properties are precisely analogous to those of the operation which we have been studying, so that all these operations are variations of a single one. The situation is briefly to be summed up as follows:

Let us consider the four pairs 1/1, 0/0, 1/0, 0/1, in their relations to one another, and to the other pairs of our system.

It is easy to show that

$$(a)\qquad \left(\frac{0}{1}\right)^2 = \frac{0}{0},$$

$$(b)\qquad \left(\frac{1}{0}\right)^2 = \frac{0}{0},$$

$$(c)\qquad \left(\frac{0}{0}\right)^2 = \frac{1}{1},$$

$$(d)\qquad \left(\frac{1}{1}\right)^2 = \frac{1}{1},$$

$$(e)\qquad \frac{1}{1} \circ \frac{0}{0} = \frac{0}{0},$$

$$(f)\qquad \frac{1}{0} \circ \frac{0}{1} = \frac{1}{1},$$

$$(g)\qquad \left(\frac{1}{0}\right)^3 = \frac{0}{1}.$$

And thus the mutual relations of the four modulus elements are stated.

But when we combine a pair a/b with each of the four moduli in succession, we get the following results:

$$(1)\qquad \frac{a}{b} \circ \frac{1}{1} = \frac{a}{b} \text{ whose cube is } \frac{b}{a},$$

$$(2)\qquad \frac{a}{b} \circ \frac{0}{0} = \frac{\bar{a}}{\bar{b}} \text{ whose cube is } \frac{\bar{b}}{\bar{a}},$$

$$(3)\qquad \frac{a}{b} \circ \frac{0}{1} = \frac{\bar{b}}{a} \text{ whose cube is } \frac{a}{\bar{b}},$$

$$(4)\qquad \frac{a}{b} \circ \frac{1}{0} = \frac{b}{\bar{a}} \text{ whose cube is } \frac{\bar{a}}{b}.$$

If, hereupon, we ask what directed pairs can be formed from a given pair a/b, by considering the four Boolean elements a, $\bar{a}$, b, $\bar{b}$, and by treating their various pairs as directed pairs, we see that the foregoing table of eight directed pairs contains all of the possible combinations, and shows how all the eight can be formed from any one of their number by using the two operations of applying the four moduli, and of raising to the third power.

But the processes in question can be greatly simplified by considering that all the four moduli can be derived from a single one of their number, by merely using our ordinal pair operation. The modulus chosen for this purpose may be either 0/1 or 1/0, at

pleasure. Thus, if we begin with 0/1, we derive the other moduli simply by considering the powers of 0/1. For we have:

$$\left(\frac{0}{1}\right)^2=\frac{0}{0};\quad \left(\frac{0}{1}\right)^3=\frac{1}{0};\quad \left(\frac{0}{1}\right)^4=\frac{1}{1}.$$

Starting with any pair a/b, and with the single modulus 0/1, we can therefore form all the derivative pairs $\bar{a}/b$, $a/\bar{b}$, etc., merely by repeating the processes of combining with the modulus, and of raising to powers.

It is possible, however, to define a new operation such that one of the moduli, say 0/1, is the unit pair of this operation. The latter will then be derived from (and in essence equivalent to) our present ordinal pair-operation. Let us use $\cup$ as the symbol of the new operation whereof 0/1 is to be the unit pair. That is, let us require an ordinal pair-operation $\cup$ to be defined such that $a/b \cup 0/1 = a/b$ whatever pair a/b may be. To this end we have only to define $\cup$ by the equation

$$\frac{a}{b}\cup\frac{c}{d}=\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{1}{0}.$$

For then

$$\frac{a}{b}\cup\frac{0}{1}=\frac{a}{b}\circ\frac{0}{1}\circ\frac{1}{0}=\frac{a}{b}\circ\frac{1}{1}=\frac{a}{b}.$$

The new operation will be so related to the old that, if $a/b \circ c/d = g/h$, then, by definition, $a/b \cup c/d = g/h \circ 1/0 = h/g$.

It is plain that by the use of the modulus element, and by raising to powers, all the results of the new operation $\cup$ can be stated in terms of our foregoing operation $\circ$, and conversely. The only novelty of the operation $\cup$ will, therefore, depend upon its choice of one of the modulus elements as its special unit.

The four moduli of our system of directed pairs are themselves pairs, and are not ordinary Boolean elements. They serve to give to the whole system properties that I believe to be not only of interest in themselves, but of no small promise for the future. In any case, here is a definite extension of the Boolean calculus, and a definite and new introduction of group-theory into this realm of the algebra of logic.

I must leave to later papers the lessons to which our study points the way. They will concern questions which I believe to be of wide philosophical bearing.

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